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Optimal convergence properties of the FETI domain decomposition method

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SUMMARY

In this paper an original variant of the FETI domain decomposition method is introduced for heterogeneous media. This method uses new absorbing interface conditions in place of the Neumann interface conditions defined in the classical FETI method. The optimal convergence properties of the classical FETI method and of its variant are first demonstrated, both in the case of homogeneous and heterogeneous media. Secondly, novel and efficient absorbing interface conditions, which avoid rigid body motions, are investigated and analysed. Numerical experiments illustrate the dependence of the proposed method upon several parameters, and confirm the robustness and efficiency of this method when equipped with such absorbing interface conditions. Copyright © 2006 John Wiley & Sons, Ltd.

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1. INTRODUCTION

These last ten years, several research have been carried out on non-overlapping domain decomposition methods [1–3]. In these methods, the initial domain is partitioned into small non-overlapping sub-domains. The continuity is enforced by using some primal or dual unknowns defined on the interface between the sub-domains. A very powerful and efficient member of this class of domain decomposition methods is the finite element tearing and interconnecting (FETI)

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method [4]. In its original version, a Neumann problem is solved on each sub-domain. Later on, in a variant of this method [4], an additional Dirichlet problem is solved exactly on each sub-domain. Unfortunately, for the Laplace equation, when dealing with arbitrary mesh partitioning, it may appear that one or more sub-domains are not attached to an external boundary condition. This leads to so-called floating sub-domains, and to non-well-posed Neumann sub-problems. An additional procedure for the detection of the rigid body motions is thus mandatory to compute the solution inside these floating sub-domains [5]. Incidently, this extra work for the detection can be cast in a coarse grid preconditioner that allows to give the scalability of the method [4]. For the Helmholtz equation, when using the FETI method, local resonance frequencies may appear in some sub-domains. A remedy to this problem, consists of defining a relative fine coarse grid, similar to the case of floating sub-domains for the Laplace equation. Another solution consists of changing the Neumann interface conditions by Robin interface conditions, with complex coefficients, as first introduced in Reference [6]. Using complex coefficients shifts the eigenvalues of the partial differential operator to the complex plane, and avoids any local resonance frequencies [7–11].

In this paper an original variant of the FETI domain decomposition method is introduced for heterogeneous media. This method uses *new real absorbing interface conditions* in place of the Neumann interface conditions defined in the classical FETI method. The optimal convergence properties of the classical FETI method and of its variant are then demonstrated, both in the case of homogeneous and heterogeneous media. The design of novel and efficient absorbing interface conditions is then investigated, derived and analysed.

The structure of this paper is the following. Section 2 presents an analysis in the Fourier space of some interface conditions. Dirichlet interface conditions are first introduced in Section 2.1 in the context of the Schur method. Mixed type interface conditions are introduced in Section 2.2 in the context of the FETI method. Section 2.3 presents the optimal convergence properties of the FETI method with Neumann and with mixed type interface conditions, both in the case of homogeneous and heterogeneous media. In Section 3, a discrete analysis defines the optimal choice of the discrete absorbing interface conditions. Section 4 discusses the design of suitable real absorbing interface conditions. Section 5 shows some numerical experiments and illustrates the convergence of the original variant of the FETI method equipped with these real absorbing interface conditions. Finally, in Section 6 the conclusion of this paper is presented.

2. FOURIER ANALYSIS OF INTERFACE CONDITIONS

In this section, the problem $-\nabla(\mu\nabla u) = f(x, y)$, $x, y \in \Omega$ is considered in the domain $\Omega = \mathbb{R}^2$ with homogeneous Dirichlet boundary conditions at infinity, i.e. $\lim_{r\to\infty} u = 0$ where $r = \sqrt{x^2 + y^2}$. The domain Ω is partitioned into two non-overlapping sub-domains $\Omega^{(1)} = (-\infty, 0] \times \mathbb{R}$ and $\Omega^{(2)} = [0, \infty) \times \mathbb{R}$, with an interface $\Gamma = \{0\} \times \mathbb{R}$. For the sake of simplicity the coefficient μ is assumed to be constant per sub-domain.

2.1. Case of Dirichlet interface conditions

The Schur method is first considered, see, e.g. Reference [12]. This method consists in assuming that the solution is known on the interface, i.e. that $u^{(1)}(0, y) = u^{(2)}(0, y) = \lambda$ with λ as a known

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value. The solution of the subproblems with Dirichlet interface conditions

$$-\nabla(\mu^{(1)}\nabla u^{(1)}) = f^{(1)}, \quad x<0, \quad y\in\mathbb{R} \tag{1}$$

$$u^{(1)} = \lambda, \quad x=0, \quad y\in\mathbb{R} \tag{2}$$

$$-\nabla(\mu^{(2)}\nabla u^{(2)}) = f^{(2)}, \quad x>0, \quad y\in\mathbb{R} \tag{3}$$

$$u^{(2)} = \lambda, \quad x=0, \quad y\in\mathbb{R} \tag{4}$$

gives $u^{(1)}(x, y) = u^{(1)}(x, y; \lambda, f^{(1)})$ and $u^{(2)}(x, y) = u^{(2)}(x, y; \lambda, f^{(2)})$. The continuity condition

$$\mu^{(1)}\partial_x u^{(1)} = \mu^{(2)}\partial_x u^{(2)}, \quad x=0, \quad y\in\mathbb{R} \tag{5}$$

must be imposed in order to have the equivalence of the subproblems (1)–(4) with the global problem. After substitution of $u^{(1)}(x, y; \lambda, f^{(1)})$ and $u^{(2)}(x, y; \lambda, f^{(2)})$ in the continuity conditions and taking into account the linearity, a condensed interface system upon the variable λ is obtained. The solution of this system gives λ and thus the value $u^{(1)}$, respectively, $u^{(2)}$ can be obtained from the solution of (1)–(2), respectively, (3)–(4).

In order to analyse the interface conditions, it suffices to consider by linearity the case $f(x, y) = 0$. The Fourier transform is used in the y direction defined for a function g by

$$\hat{g}(x, k) = \int_{-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}ky} g(x, y)\,\mathrm{d}y$$

and applied to the systems of Equations (1)–(4). Since the coefficient μ is assumed to be constant per sub-domain, this leads to

$$-\mu^{(1)}\partial_{xx}\hat{u}^{(1)} + \mu^{(1)}k^2\hat{u}^{(1)} = 0, \quad x<0, \quad k\in\mathbb{R} \tag{6}$$

$$\hat{u}^{(1)} = \hat{\lambda}, \quad x=0, \quad k\in\mathbb{R} \tag{7}$$

$$-\mu^{(2)}\partial_{xx}\hat{u}^{(2)} + \mu^{(2)}k^2\hat{u}^{(2)} = 0, \quad x>0, \quad k\in\mathbb{R} \tag{8}$$

$$\hat{u}^{(2)} = \hat{\lambda}, \quad x=0, \quad k\in\mathbb{R} \tag{9}$$

with the continuity constraint

$$\mu^{(1)}\partial_x\hat{u}^{(1)} = \mu^{(2)}\partial_x\hat{u}^{(2)}, \quad x=0, \quad y\in\mathbb{R} \tag{10}$$

The general solution of these ordinary differential equations is

$$\hat{u}^{(s)} = C\,\mathrm{e}^{|k|x} + B\,\mathrm{e}^{-|k|x}, \quad s=1,2$$

Since the Dirichlet boundary condition excludes growing solutions at infinity the following solutions are obtained:

$$\hat{u}^{(1)}(x, k) = C\,\mathrm{e}^{|k|x}, \quad \hat{u}^{(2)}(x, k) = B\,\mathrm{e}^{-|k|x} \tag{11}$$

After substitution of (11) in the interface conditions (7), (9) and evaluation at $x=0$, this leads to: $C=\hat{\lambda}$ and $B=\hat{\lambda}$. Using the continuity constraint (10) and the fact that

$$\partial_x \hat{u}^{(1)}(x,k)=|k|\hat{u}^{(1)}(x,k), \quad \partial_x \hat{u}^{(2)}(x,k)=-|k|\hat{u}^{(2)}(x,k)$$

leads after substitution to a linear system upon the unknown $\hat{\lambda}$

$$(\mu^{(1)}|k|+\mu^{(2)}|k|)\hat{\lambda}=0$$

which can be written in the form $\mathscr{S}\hat{\lambda}=0$ with $\mathscr{S}=(\mathscr{S}^{(1)}+\mathscr{S}^{(2)})$ where $\mathscr{S}^{(1)}=\mu^{(1)}|k|$ and $\mathscr{S}^{(2)}=\mu^{(2)}|k|$. This condensed linear system on the interface has one and only one solution $\hat{\lambda}$ equal to zero if and only if $\mathscr{S}$ is invertible.

2.2. *Case of absorbing interface conditions*

The FETI method with absorbing interface conditions is now considered [9]. If $\mathscr{A}$ denotes an appropriate linear operator and if λ is assumed to be given, the solution of the subproblems with absorbing interface conditions

$$-\nabla(\mu^{(1)}\nabla u^{(1)})=f^{(1)}, \quad x<0, \quad y\in\mathbb{R} \tag{12}$$

$$(\mu^{(1)}\partial_x+\mathscr{A})u^{(1)}=\lambda, \quad x=0, \quad y\in\mathbb{R} \tag{13}$$

$$-\nabla(\mu^{(2)}\nabla u^{(2)})=f^{(2)}, \quad x>0, \quad y\in\mathbb{R} \tag{14}$$

$$(\mu^{(2)}\partial_x+\mathscr{A})u^{(2)}=\lambda, \quad x=0, \quad y\in\mathbb{R} \tag{15}$$

gives $u^{(1)}(x,y)=\tilde{u}^{(1)}(x,y;\lambda,f^{(1)})$ and $u^{(2)}(x,y)=\tilde{u}^{(2)}(x,y;\lambda,f^{(2)})$. The continuity condition

$$u^{(1)}=u^{(2)}, \quad x=0, \quad y\in\mathbb{R}$$

must be imposed in order to have the equivalence of the sub-problems with the global problem. After substitution and linearity a condensed interface system upon the variable λ is obtained. The solution to this system gives the correct value of λ and thus the values $u^{(1)}$ and $u^{(2)}$ can be obtained. Note that for the classical FETI method $\mathscr{A}$ is equal to zero, and without the detection of the rigid body motions the method does not ensure well-posed subproblems for general mesh partitioning [5].

Following the same lines as in the previous section, after applying the Fourier transform to the systems of Equations (12)–(15), leads to

$$-\mu^{(1)}\partial_{xx}\hat{u}^{(1)}+\mu^{(1)}k^2\hat{u}^{(1)}=0, \quad x<0, \quad k\in\mathbb{R} \tag{16}$$

$$(\mu^{(1)}\partial_x+\sigma(k))u^{(1)}=\hat{\lambda}, \quad x=0, \quad k\in\mathbb{R} \tag{17}$$

$$-\mu^{(2)}\partial_{xx}\hat{u}^{(2)}+\mu^{(2)}k^2\hat{u}^{(2)}=0, \quad x>0, \quad k\in\mathbb{R} \tag{18}$$

$$(\mu^{(2)}\partial_x+\sigma(k))u^{(2)}=\hat{\lambda}, \quad x=0, \quad k\in\mathbb{R} \tag{19}$$

with the continuity constraint

$$\hat{u}^{(1)} = \hat{u}^{(2)}, \quad x = 0, \quad y \in \mathbb{R} \tag{20}$$

where $\sigma(k)$ denotes the Fourier symbol of the operator $\mathscr{A}$. Taking into account the Dirichlet boundary conditions at infinity, the solutions are:

$$\hat{u}^{(1)}(x,k) = C\,\mathrm{e}^{|k|x}, \quad \hat{u}^{(2)}(x,k) = B\,\mathrm{e}^{-|k|x} \tag{21}$$

Using these interface conditions (21) in Equation (17), (19) and after evaluating at $x = 0$ leads to

$$(\mu^{(1)}|k| + \sigma(k))C = \hat{\lambda}, \quad (-\mu^{(2)}|k| + \sigma(k))B = \hat{\lambda}$$

The value of C and B can be obtained

$$C = \frac{\hat{\lambda}}{\mu^{(1)}|k| + \sigma(k)} \quad \text{and} \quad B = \frac{\hat{\lambda}}{-\mu^{(2)}|k| + \sigma(k)}$$

and this gives by induction

$$\hat{u}^{(1)}(x,k) = \frac{\hat{\lambda}}{\mu^{(1)}|k| + \sigma(k)}\,\mathrm{e}^{|k|x} \quad \text{and} \quad \hat{u}^{(2)}(x,k) = \frac{\hat{\lambda}}{-\mu^{(2)}|k| + \sigma(k)}\,\mathrm{e}^{-|k|x}$$

Using the continuity constraint (20) leads after substitution to a linear system upon the unknown $\hat{\lambda}$

$$\left(\frac{1}{\mu^{(1)}|k| + \sigma(k)} - \frac{1}{-\mu^{(2)}|k| + \sigma(k)}\right)\hat{\lambda} = 0$$

which can be written as: $\mathscr{D}\hat{\lambda} = 0$ with $D = (\mathscr{D}^{(1)} + \mathscr{D}^{(2)})$ where

$$\mathscr{D}^{(1)} = \frac{1}{\mu^{(1)}|k| + \sigma(k)} \quad \text{and} \quad \mathscr{D}^{(2)} = \frac{-1}{-\mu^{(2)}|k| + \sigma(k)}$$

This condensed linear system on the interface has one and only one solution $\hat{\lambda}$ equal to zero if and only if $\mathscr{D}$ is invertible.

2.3. *Optimal interface conditions in the Fourier space*

In the previous analysis the condensed linear system on the interface derived from the Schur method with Dirichlet interface conditions and from the FETI method with absorbing interface conditions, have been derived in the Fourier transformed space. When a right-hand side is considered, the solution of this condensed linear system gives $\hat{\lambda}$ and the solution $\hat{u}^{(s)}$ in each sub-domain can then be obtained. It can be shown [4, 12] that the spectrum of the eigenvalues of the operator $\mathscr{D}$ of the classical FETI method is strongly different from the spectrum of the eigenvalues of the operator $\mathscr{S}$ of the Schur method. As a consequence, the classical FETI method is more efficient than the Schur method and is commonly used. In order to improve the efficiency of the classical FETI method, an

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additional Dirichlet problem can be solved in each sub-domain [4]. This preconditioning technique consists in using the Schur method as a left preconditioner of the classical FETI method. In the case of the FETI method with absorbing interface conditions, an extension of this preconditioning technique can be obtained and leads in the Fourier transformed space to the new theorem.

Theorem 2.1 (*Heterogeneous case*)
In the case of two sub-domains splitting, and with the choice $\sigma(k) = (\mu^{(2)}|k| - \mu^{(1)}|k|)/2$, the FETI method with absorbing interface conditions, preconditioned by the Dirichlet preconditioner converges in one iteration at most.

Proof
The FETI method with absorbing interface conditions preconditioned by the Dirichlet preconditioner leads to an operator of the condensed linear system on the interface equal to $\mathscr{S}\mathscr{D}$, where $\mathscr{S}$ and $\mathscr{D}$ have been derived in the previous sections. Choosing $\sigma(k)$ as defined in the previous theorem, leads to the following expression of $\mathscr{D}$ and $\mathscr{S}$:

$$\mathscr{D} = \left(\frac{1}{\mu^{(1)}|k| + \sigma(k)} - \frac{1}{-\mu^{(2)}|k| + \sigma(k)} \right) = \frac{4}{(\mu^{(1)} + \mu^{(2)})|k|}$$

$$\mathscr{S} = (\mu^{(1)} + \mu^{(2)})|k| = 4\mathscr{D}^{-1}$$

and so the operator of the preconditioned condensed linear system reduces to identity. □

In the case of an homogeneous media Ω, i.e. when μ is assumed to be constant, the problem reduces to $-\Delta u = f$. The solution of this problem with the Schur method and with the FETI method, followed by a Fourier analysis leads to the corollary already introduced in Reference [4].

Corollary 2.1 (*Homogeneous case*)
In the case of two sub-domains splitting, and with the choice $\sigma(k) = 0$, the classical FETI method preconditioned by the Dirichlet preconditioner converges in one iteration at most.

Proof
Like in Theorem 2.1, a choice of $\sigma(k)$ equal to zero leads to the following expression of the operators $\mathscr{D}$ and $\mathscr{S}$

$$\mathscr{D} = \left(\frac{1}{\mu|k|} - \frac{1}{-\mu|k|} \right) = \frac{2}{\mu|k|} \quad \text{and} \quad \mathscr{S} = 2\mu|k| = 2\mathscr{D}^{-1}$$

and so far the operator $\mathscr{S}\mathscr{D}$ of the preconditioned condensed linear system reduces to identity. □

3. DISCRETE ANALYSIS OF INTERFACE CONDITIONS

In this section, the domain $\Omega \in \mathbb{R}^d$, $d = 1, 2, 3$, is split into two non-overlapping sub-domains $\Omega^{(1)}$ and $\Omega^{(2)}$ with an interface Γ. Considering that subscripts i and p denotes the degrees of freedom located inside sub-domain $\Omega^{(s)}$ and on the interface Γ then the contribution of sub-domain $\Omega^{(s)}$,

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$s=1,2$ to the sub-domain matrix and the right-hand side can be written as

$$K^{(s)}=\begin{pmatrix} K_{ii}^{(s)} & K_{ip}^{(s)} \\ K_{pi}^{(s)} & K_{pp}^{(s)} \end{pmatrix}, \quad b^{(s)}=\begin{pmatrix} b_i^{(s)} \\ b_p^{(s)} \end{pmatrix}$$

In the following it is assumed that matrix $K_{ii}^{(s)}$ is non-singular for $s=1,2$. The global problem is a block system obtained by assembling local contribution of each sub-domain

$$\begin{pmatrix} K_{ii}^{(1)} & 0 & K_{ip}^{(1)} \\ 0 & K_{ii}^{(2)} & K_{ip}^{(2)} \\ K_{pi}^{(1)} & K_{pi}^{(2)} & K_{pp} \end{pmatrix} \begin{pmatrix} x_i^{(1)} \\ x_i^{(2)} \\ x_p \end{pmatrix} = \begin{pmatrix} b_i^{(1)} \\ b_i^{(2)} \\ b_p \end{pmatrix} \tag{22}$$

The matrices $K_{pp}^{(1)}$ and $K_{pp}^{(2)}$ represent the interaction matrices between the nodes on the interface obtained by integration on $\Omega^{(1)}$ and on $\Omega^{(2)}$. Block K_{pp} is the sum of these two blocks. In the same way the term $b_p=b_p^{(1)}+b_p^{(2)}$ is obtained by local integration of the right-hand side over each sub-domain and summation on the interface. It can be shown that the global problem (22) is equivalent to the following coupled subproblems:

$$\begin{pmatrix} K_{ii}^{(1)} & K_{ip}^{(1)} \\ K_{pi}^{(1)} & K_{pp}^{(1)}+A_{pp} \end{pmatrix} \begin{pmatrix} x_i^{(1)} \\ x_p^{(1)} \end{pmatrix} = \begin{pmatrix} b_i^{(1)} \\ b_p^{(1)}+\lambda \end{pmatrix} \tag{23}$$

$$\begin{pmatrix} K_{ii}^{(2)} & K_{ip}^{(2)} \\ K_{pi}^{(2)} & K_{pp}^{(2)}-A_{pp} \end{pmatrix} \begin{pmatrix} x_i^{(2)} \\ x_p^{(2)} \end{pmatrix} = \begin{pmatrix} b_i^{(2)} \\ b_p^{(2)}-\lambda \end{pmatrix} \tag{24}$$

with the coupling equations

$$x_p^{(1)}=x_p^{(2)} \tag{25}$$

where λ denotes an additional unknown defined on the interface.

As detailed in Section 4, the matrix A_{pp} defined Equations (23) and (24) corresponds to the discretization of the continuous operator $\mathscr{A}$ defined Equations (13) and (15).

3.1. Case of discrete Dirichlet interface conditions

Similar to the continuous analysis, one way to solve the subproblems (23)–(25), is to assume that the continuity condition $x_p^{(1)}=x_p^{(2)}=x_p$ is satisfied, with x_p being an arbitrary known value. After elimination of $x_i^{(1)}$ and $x_i^{(2)}$ in favour of x_p inside (23) and (24) the following is obtained:

$$[S^{(1)}+A_{pp}]x_p=c_p^{(1)}+\lambda, \quad [S^{(2)}-A_{pp}]x_p=c_p^{(2)}-\lambda$$

where $S^{(s)}=K_{pp}^{(s)}-K_{pi}^{(s)}[K_{ii}^{(s)}]^{-1}K_{ip}^{(s)}$ is the Schur complement matrix and $c_p^{(s)}=b_p^{(s)}-K_{pi}^{(s)}[K_{ii}^{(s)}]^{-1}b_i^{(s)}$ is the condensed right-hand side in sub-domain $\Omega^{(s)}$. After addition of the two previous

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equations the following linear system is obtained:

$$(S^{(1)} + S^{(2)})x_p = c_p^{(1)} + c_p^{(2)}$$

The solution of this condensed linear system on the interface leads to the unique correct value of x_p, and the first line of (23) and (24)

$$x_i^{(1)} = [K_{ii}^{(1)}]^{-1}(b_i^{(1)} - K_{ip}^{(1)} x_p^{(1)}), \quad x_i^{(2)} = [K_{ii}^{(2)}]^{-1}(b_i^{(2)} - K_{ip}^{(2)} x_p^{(2)})$$

gives, respectively, the values of $x_i^{(1)}$ and $x_i^{(2)}$.

3.2. *Case of discrete absorbing interface conditions*

Similar to the continuous analysis, the FETI method with Neumann interface conditions, i.e. with $A_{pp} = 0$, consists of supposing that λ is a known value. With the choice $A_{pp} = 0$ the subproblems (23), (24) may not be well-posed. In this case, the FETI method with absorbing interface conditions, i.e. with $A_{pp} \neq 0$ should be considered. The matrix A_{pp} can be chosen in such a way that the singularities disappear in the local subproblems. A direct relation between $x_p^{(s)}$, for $s = 1, 2$ and λ can be obtained from (23) and (24) and leads to

$$x_p^{(1)} = [S^{(1)} + A_{pp}]^{-1}(c_p^{(1)} + \lambda), \quad x_p^{(2)} = [S^{(2)} - A_{pp}]^{-1}(c_p^{(2)} - \lambda)$$

Then substitution in (25) gives

$$([S^{(1)} + A_{pp}]^{-1} + [S^{(2)} - A_{pp}]^{-1})\lambda = -[S^{(1)} + A_{pp}]^{-1} c_p^{(1)} + [S^{(2)} - A_{pp}]^{-1} c_p^{(2)}$$

The solution of this system gives the only convenient value of λ and then $x_i^{(1)}$, $x_p^{(1)}$, $x_i^{(2)}$ and $x_p^{(2)}$ can be obtained. Using such absorbing interface conditions can be seen as a nice alternative to the detection of the rigid body motions and to the use of pseudo-inverses in the classical FETI method [5, 4].

3.3. *Optimal interface conditions in the discrete space*

Previously, the Schur method, the classical FETI method and its variant have been presented with a discrete analysis. A theorem is now introduced in the discrete space:

Theorem 3.1
In the case of two sub-domains splitting, and with the choice of the matrix $A_{pp} = \frac{1}{2}(S^{(2)} - S^{(1)})$ the FETI method with absorbing interface conditions preconditioned by the Dirichlet preconditioner converges in one iteration at most.

Proof
In the particular case where $S^{(1)} = S^{(2)}$, A_{pp} is equal to zero and the theorem reduces to the theorem introduced by Farhat *et al.* [4]. When $S^{(1)} \neq S^{(2)}$, it has been shown in the previous sections that the FETI method with absorbing interface conditions leads to the linear system

$$([S^{(1)} + A_{pp}]^{-1} + [S^{(2)} - A_{pp}]^{-1})\lambda = -[S^{(1)} + A_{pp}]^{-1} c_p^{(1)} + [S^{(2)} - A_{pp}]^{-1} c_p^{(2)}$$

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Choosing the matrix A_{pp} equal to: $A_{pp} = \frac{1}{2}(S^{(2)} - S^{(1)})$ transforms the left-hand side of this linear system to

$$([S^{(1)} + A_{pp}]^{-1} + [S^{(2)} - A_{pp}]^{-1})\lambda = ([S^{(1)} + \tfrac{1}{2}(S^{(2)} - S^{(1)})]^{-1} + [S^{(2)} - \tfrac{1}{2}(S^{(2)} - S^{(1)})]^{-1})\lambda$$
$$= (4[S^{(1)} + S^{(2)}]^{-1})\lambda$$

and the right-hand side becomes

$$-[S^{(1)} + A_{pp}]^{-1}c_p^{(1)} + [S^{(2)} - A_{pp}]^{-1}c_p^{(2)}$$
$$= -[S^{(1)} + \tfrac{1}{2}(S^{(2)} - S^{(1)})]^{-1}c_p^{(1)} + [S^{(2)} - \tfrac{1}{2}(S^{(2)} - S^{(1)})]^{-1}c_p^{(2)}$$
$$= (2[S^{(1)} + S^{(2)}]^{-1})(-c_p^{(1)} + c_p^{(2)})$$

As already explained, preconditioning the FETI method with the Dirichlet preconditioner correspond to a left multiplication of the linear system of the FETI method by the matrix $[S^{(1)} + S^{(2)}]$. So far, the preconditioned system reduces to identity, which concludes the proof. □

4. DESIGN OF REAL ABSORBING INTERFACE CONDITIONS

4.1. Preliminaries

For the Helmholtz equation with a wave number ω, i.e. $(-\Delta - \omega^2)u = f$, a general mesh partitioning may introduce some resonance frequencies in some sub-domains. In order to avoid such resonance frequencies Farhat *et al.* [9] have proposed to define the operator $\mathscr{A}$ equal to $\mathrm{i}\omega$, where $\mathrm{i} = \sqrt{-1}$ denotes the imaginary complex number. Using such a complex operator shifts the eigenvalues of the Δ-operator to the complex plane, and avoids any local resonance frequencies [6–8, 10, 13–16]. An additional benefit of these interface conditions consists of the local properties of the operators as investigated for localized non-reflecting absorbing conditions, see Reference [17] for instance. Using such a complex operator was natural for exterior Helmholtz problems since the radiation condition at infinity involves complex coefficients. In this paper, the equation $-\nabla(\mu\nabla u) = f$, in heterogeneous media of density μ is considered, and the boundary conditions are real. So far *the operator* $\mathscr{A}$ *is assumed to be real*. This is the key point of the following analysis, since not considering complex operator completely change the properties of the condensed interface system.

At this point it is important to mention the previous work of Farhat *et al.* [5] for the equation of linear elasticity. The proposed idea was to compute in the discrete space directly the matrix A_{pp} associated with the operator $\mathscr{A}$. For this purpose, the matrix of the subproblem was condensed on one point of the interface, and after multiplication by a coefficient $0<\gamma<1$ was considered as the matrix A_{pp}. Unfortunately, this approach involves non-local interface operations. In addition this approach deteriorates the conditioning number of the local subproblems which slows down the convergence of the FETI method. In this paper, the matrix A_{pp} is obtained from the discretization of an operator $\mathscr{A}$ defined as a partial differential operator acting on the interface. As a consequence

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only local interface operations are performed. This partial differential operator is determined after a Fourier analysis of the FETI method with absorbing interface conditions.

4.2. Polynomial approximations

In the previous section the optimal absorbing interface conditions used with a Dirichlet preconditioner have been presented. Such optimal absorbing interface conditions involve in the Fourier transformed space the operator $(\mu^{(2)}|k| - \mu^{(1)}|k|)/2$. This operator corresponds to a non-local operator in the physical space. This means that in practice, using such optimal absorbing interface conditions will imply a high computational cost.

In this section two choices for the operator $\sigma(k)$ are investigated and analysed. The proposed choices are based on different approximations of the operator $\mu|k|$ which corresponds to a non-local operator in the physical space, by polynomials $\sigma(k)$ which represent differential operators in the physical space and are thus local. To avoid an increase in the bandwidth of the local subproblems, polynomials of degree at most two are considered, which leads to an interface operator $\mathscr{A}$ which is at most a second-order partial differential operator acting along the interface. By symmetry of the equation there is no interest in a first-order term. Therefore the operator $\mu|k|$ is approximated either by a polynomial of degree zero, i.e. a constant, or by a polynomial of degree two. The first approximation leads to zeroth-order interface condition and the second approximation leads to second-order interface condition.

In the Fourier analysis, the inverse of the quantity $\mu|k| \pm \sigma(k)$ appears in the expression of the linear system condensed on the interface. In order to ensure the existence and uniqueness of the solution of this linear system, the quantity $\mu|k| \pm \sigma(k)$ must not vanish, for all k. However, in practice only the frequencies carried out by the numerical grid are of interest.

The approximation of the operator $\mu|k|$ by a polynomial $\sigma(k)$ of degree zero, i.e. a constant is first considered. Since the graph of $\mu|k|$ must not intersect the graph of $\sigma(k)$ in the frequency range $[-k_{\max}, +k_{\max}]$, any value of $\alpha > \max(\mu^{(1)}, \mu^{(2)})k_{\max}$ is convenient. In this paper, the simplest choice $\alpha = \max(\mu^{(1)}, \mu^{(2)})k_{\max} + \delta$ is considered, where δ denotes a very small number linked with the precision of the computer. This choice of $\sigma(k) = \alpha$ in the transformed Fourier space leads to the operator $\mathscr{A}u = \alpha u$ in the physical space. The absorbing interface condition defined on the interface of the sub-domains takes the form $\mu(\partial u/\partial n) \pm \alpha u = \pm\lambda$ where n denotes the unitary normal derivative vector along the interface.

The approximation of the operator $\mu|k|$ by a polynomial $\sigma(k)$ of degree two is now considered. Since the graph of $\mu|k|$ must not intersect the graph of $\sigma(k)$ in the frequency range $[-k_{\max}, +k_{\max}]$, any expression $\sigma(k) = \alpha + \beta k^2$, with $\alpha > \max(\mu^{(1)}, \mu^{(2)})k_{\max}$ and with $\beta > 0$ is convenient. This choice of $\sigma(k) = \alpha + \beta k^2$ in the transformed Fourier space leads to the operator $\mathscr{A}u = \alpha u + \beta\partial^2_{\tau\tau}u$ in the physical space. The absorbing interface condition defined on the interface of the sub-domains takes the form $\mu(\partial u/\partial n) \pm (\alpha u + \beta(\partial^2 u/\partial\tau^2)) = \pm\lambda$ where n denotes the unitary normal derivative vector along the interface and where τ denotes the tangent direction at the interface. A finer determination of the coefficients α and β can be obtained in the particular two-dimensional case with an uniform triangular mesh of mesh size h. The coefficients of the elementary volume stiffness matrix K^{e}, of the elementary surface mass matrix M^{e}_{Γ} and of the elementary surface stiffness matrix K^{e}_{Γ} are, respectively, in order $O(1)$, $O(h)$ and $O(h^{-1})$. So far are the coefficients of the matrix K^{e}_{pp}, which denotes the restriction to a edge of the elementary stiffness matrix. In order to keep the homogeneity of the dimension, the elementary matrix $A^{\mathrm{e}}_{pp} = \alpha M^{\mathrm{e}}_{\Gamma} + \beta K^{\mathrm{e}}_{\Gamma}$, added to the elementary matrix K^{e}_{pp}, should be in $O(1)$. This is realized when the coefficients α and β

are chosen, respectively, in order $O(h^{-1})$ and in order $O(h)$. In this paper, the simplest choice of α and β are considered, i.e. $\alpha = \max(\mu^{(1)}, \mu^{(2)})h^{-1}$ and $\beta = \max(\mu^{(1)}, \mu^{(2)})h$. The reader can check that with this choice the operator $\sigma(k) = \alpha + \beta k^2$ does not intersect the graph of the operator $\mu|k|$, and as a consequence the subproblems are well posed in each sub-domain.

Although a simple choice for the coefficients α and β in the localized absorbing interface condition is given in the case of a structured mesh with a regular interface, the generalization of this choice to unstructured non-uniform triangulations does not lead to particular difficulties. In this case, an average mesh size h is evaluated, and the previous choice of the coefficients α and β is applied on each edge of the interface.

Remark 4.1

The discrete interface conditions proposed in this paper are local. The idea is to use polynomial approximations for the Fourier transform of the linear operator $\mathscr{A}$, which translates to localized differential (tangential to the interface) operators in the physical space. After discretization, the matrix A_{pp} has the same bandwidth than the local matrix.

5. NUMERICAL EXPERIMENTS

In order to illustrate the domain decomposition method presented in this paper, four experiments are now presented. The problem considered is

$$\begin{aligned} -\nabla(\mu\nabla u) &= 0, && \forall (x, y) \in]0, 1[\times]0, 1[\\ u(x, y) &= u_0(x, y), && x = 0,\ y \in [0, 1] \\ \partial_x u(x, y) &= \partial_x u_0(x, y), && x = 1,\ y \in [0, 1] \\ \partial_y u(x, y) &= \partial_y u_0(x, y), && x \in [0, 1],\ y = 0, 1 \end{aligned}$$

where $u_0(x, y) = 16((2x-1)^2 - (2y-1)^2)$. Unstructured non-uniform triangulations are generated from unstructured uniform triangulations by shifting the coordinates of the nodes with a random value. An example is illustrated in Figure 1. The global domain is first split into two rectangular shaped sub-domains $\Omega^{(1)} = [0, 0.5] \times [0, 1]$ and $\Omega^{(2)} = [0.5, 1] \times [0, 1]$ as shown Figure 1. In this configuration external Dirichlet boundary conditions are only defined on the boundary of the sub-domain $\Omega^{(1)}$. As a consequence the matrix in the sub-domain $\Omega^{(2)}$ is singular, and the FETI method with Neumann interface conditions cannot converge without an additional procedure of the detection of the rigid body motions. The number of iterations required by the different methods are reported in Table I for different density assumed constant per sub-domain. It can be noticed that the FETI method with second-order absorbing interface conditions converges faster than the other methods. The respective performance of the methods can be classified as follows: first, the FETI method with second-order absorbing interface conditions, second the FETI method with Neumann interface conditions, third the Schur method with Dirichlet interface conditions, fourth the FETI method with zeroth-order conditions. These respective performance will be confirmed in Tables I and II. Table I clearly illustrates the robustness of the FETI method with second-order absorbing interface conditions and its supremacy over the other methods, specially when the

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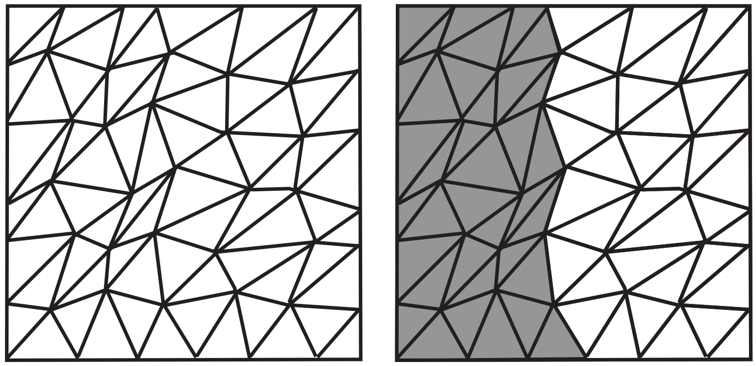

Figure 1. Example of a non-uniform triangulation (left) and example of a mesh partitioning into two sub-domains (right).

Table I. Number of iterations upon the density assumed constant per sub-domain ($N_s = 2$, $h = 1/100$).

Density ratio $\mu^{(2)}/\mu^{(1)}$	Schur with Dirichlet interface conditions	FETI* with Neumann interface conditions	FETI with zeroth-order interface conditions	FETI with second-order interface conditions
10^0	35	16	71	16
10^1	36	23	71	19
10^2	39	26	72	21
10^3	40	30	74	22

*An additional procedure for the detection of the rigid body motions has been incorporated.

Table II. Number of iterations for different mesh size parameter ($N_s = 2$, $\mu^{(1)} = \mu^{(2)} = 1$).

Mesh size h	Schur with Dirichlet interface conditions	FETI* with Neumann interface conditions	FETI with zeroth-order interface conditions	FETI with second-order interface conditions
1/50	24	12	34	13
1/100	35	16	71	16
1/200	51	21	147	23
1/400	73	28	262	33

*An additional procedure for the detection of the rigid body motions has been incorporated.

heterogeneity between the two sub-domains increases. This method even bits the FETI method with Neumann interface conditions equipped with the procedure of the detection of the rigid body motions! The dependency upon the mesh size parameter is now analysed and the results are reported Table II in the case of homogeneous media. Table III presents the number of iterations in the case of heterogeneous media, when each sub-domain is composed of an homogeneous media of constant density. Different configurations are finally considered. For each configuration, each sub-domain is composed only with one material. According to this hypothesis, the global

Table III. Number of iterations for different mesh size parameter ($N_s = 2$, $\mu^{(1)} = 1$, $\mu^{(2)} = 10^2$).

Mesh size h	Schur with Dirichlet interface conditions	FETI* with Neumann interface conditions	FETI with zeroth-order interface conditions	FETI with second-order interface conditions
1/50	25	18	36	16
1/100	39	26	72	21
1/200	57	36	147	29
1/400	81	55	272	44

*An additional procedure for the detection of the rigid body motions has been incorporated.

Table IV. Number of iterations for different configuration and different number of sub-domains ($h = 1/100$, $\mu^{(2p-1)} = 1$, $\mu^{(2p)} = 10^2$, $p = 1, 2, 3, 4$).

Sub-domains N_s	Schur with Dirichlet interface conditions	FETI* with Neumann interface conditions	FETI with zeroth-order interface conditions	FETI with second-order interface conditions
2	47	25	120	24
4	84	54	302	47
8	127	87	565	73
16	212	149	1101	120

*An additional procedure for the detection of the rigid body motions has been incorporated.

domain is, respectively, split into two (respectively, four, eight and sixteen) sub-domains for the first (respectively, the second, the third, and the fourth) configuration. The results reported in Table IV illustrates the strong robustness of the FETI method with second-order interface conditions and its supremacy over the FETI method with Neumann interface conditions equipped with the procedure of the detection of the rigid body motions.

Remark 5.1
As analysed in Reference [4], one iteration of the Schur method requires similar computational time than one iteration of the FETI method with Neumann interface conditions. In addition, because the bandwidth of the matrix A_{pp} is the same than the bandwidth of the local matrix, one iteration of the FETI method with Neumann interface conditions, or with zero-order interface conditions or with second-order interface conditions requires the same computational time. For these reasons, the numerical experiments only use the number of iterations to illustrate the supremacy of the FETI method with the second-order interface conditions.

It should be noticed that despite the proposed absorbing interface conditions do not involve optimization procedures (as in our previous work for the Schwarz method [18]), these interface conditions lead to extraordinary performances.

6. CONCLUSION

In this paper an original variant of the FETI method with absorbing interface conditions has been introduced. This method avoids any floating sub-domains and guarantees the well-posedness

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properties of all the subproblems by using *real absorbing interface conditions* in place of the Neumann interface conditions defined in the classical FETI method. No procedure for the detection of the rigid body motions are required anymore to compute the solution in these subproblems, even for general mesh partitioning. The absorbing interface conditions considered in this paper use zeroth-order and second-order partial differential operators defined on the interface. Due to the property of the equation, only real coefficients are considered to design these absorbing interface conditions. Several numerical results illustrate the robustness and the efficiency of the proposed method equipped with these new interface conditions upon several parameters. The results seem very promising in the case of heterogeneous media.

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